Last time: - Grover's algorithm
- Adjoint operators

Goal today: Prove the spectral theorem

Def: Let L be a linear operator on a vector space V.
If $\lambda \in \mathbb{C}$ is a scalar, we define the λ-eigenspace of L to be the subspace

$$V_\lambda = \{ |v\rangle \in V \mid L|v\rangle = \lambda |v\rangle \}.$$

If V_λ is nonzero, then we say that λ is an eigenvalue of L.
A nonzero element of V_λ is

called an eigenvector of L (with eigenvalue λ), or we might call it an eigenstate.

E.g. Consider $L: \mathbb{C}^2 \to \mathbb{C}^2$

$$|e_1\rangle \mapsto 0$$

$$|e_2\rangle \mapsto |e_1\rangle$$

(This is the ket-bra $|e_1\rangle\langle e_2|$)

Then $V_0 = \mathbb{C}|e_1\rangle$

$V_\lambda = 0 \quad \lambda \neq 0.$

(Nonzero multiples of $|e_1\rangle$ are the only eigenvectors of L)

Def: Let $V = \mathbb{C}^n$ and let L be an operator w/ eigenspaces $\{V_\lambda\}$.

If $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$ is a direct sum decomposition, then we say L is diagonalizable.

If $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$ is an orthogonal decomposition, then we say L is unitarily diagonalizable.

So L is unitarily diagonalizable when there is an orthonormal basis consisting of eigenvectors for L.

Thm: If L is an operator on a finite dimensional complex vector space, then L has an eigenvector.

PF: Note that the function $\mathbb{C} \to \mathbb{C}$

$$t \mapsto \det(tI - L)$$

is a polynomial.
By the Fundamental Theorem of Algebra, any polynomial has a root in the complex numbers.

$\Rightarrow \exists \lambda_0 \in \mathbb{C}$ st $\det(\lambda_0 I - L) = 0$.

Thus $\lambda_0 I - L$ has nonzero kernel.
Let $v \in \ker(\lambda_0 I - L)$ be nonzero.
Equivalently, $(\lambda_0 I - L)v = 0$

$$\Rightarrow Lv = \lambda_0 v \quad .$$

So v is an eigenvector for L.

$\square$

Remark: This is about the strongest thing we can say about one operator. We can get more information if we consider multiple operators.

We say v is a simultaneous eigenvector for a set of linear operators if v is an eigenvector for each operator in the set.

We say L preserves a subspace $W \subseteq V$ if $L(W) \subseteq W$.

Lemma: Any set of (pairwise) commuting operators has a common eigenvector.

PF: Problem

Def: An operator U on $\mathbb{C}^n$ is normal if $U^*U = UU^*$.

E.g. unitary operators
self-adjoint operators
$\Rightarrow$ orthogonal projection operators

Thm: (Spectral Theorem for $\mathbb{C}^n$)
Let U be an operator on a finite dimensional inner product space V. Then U is unitarily diagonalizable iff U is normal.

Pf: $(\Rightarrow)$ $U = \sum_i \lambda_i |e_i\rangle\langle e_i|$ for some orth. basis $\{e_i\}$

so $U^*U = UU^* = \sum_i \lambda_i \lambda_i^* |e_i\rangle\langle e_i|$.

$(\Leftarrow)$

We will show by induction on the dimension of V that if U is normal, then there is an orthonormal basis of V consisting of eigenvectors of U.

Lemma: If U preserves a subspace W of V, then U^* preserves $W^\perp$.

Since U and U^* commute, they have a simultaneous eigenvector v_1.

Both U and U^* preserve $\mathbb{C}v_1$, so by the Lemma, U^* and $U^{**} = U$ preserve $W \doteq (\mathbb{C}v_1)^\perp$

Note that $U|_W$ and $U^*|_W$ are adjoint operators on W.

Clearly $U|_W$ and $U^*|_W$ commute, so $U|_W$ is a normal operator on W.

W has a smaller dimension than V, so by induction there is an orthonormal basis for W consisting of eigenvectors for U.

Combining that basis with $\frac{v_1}{\sqrt{\langle v_1 | v_1 \rangle}}$ gives the desired orthonormal basis for V.

$\square$